

LAB observables for the muon polarization in $K^+ \rightarrow \pi^+ \mu^+ \mu^-$

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We analyse the muon longitudinal polarization asymmetry Δ_{long} in the decay $K^+ \rightarrow \pi^+ \mu^+ \mu^-$. It is stressed that, since the muon helicities are not Lorentz-invariant quantities, the magnitude of Δ_{long} depends in general on the reference frame. We consider the muon helicities in the LAB system, and study the sensitivity of the longitudinal polarization asymmetry to the flavour-mixing parameters in the Standard Model for stopped and in-flight decaying K^+ . A similar analysis is carried out for the decay $K_L \rightarrow \mu^+ \mu^-$. We find that in both cases the asymmetry is diluted when increasing the energy of the decaying kaons.

I. INTRODUCTION

As has been pointed out some years ago by Savage and Wise [1], the muon polarization in the $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ decay can provide important information on the structure of weak interactions and the flavour mixing. The process is dominated by a parity-conserving contribution, arising from the exchange of one photon. Nowadays the theoretical analysis of the $K \rightarrow \pi \gamma^*$ form factor is being revisited—including unitarity corrections from $K \rightarrow \pi \pi \pi$ and chiral perturbation expansion up to $\mathcal{O}(p^6)$ [2]—in view of the recent measurement [3] of the ratio $R = \Gamma(K^+ \rightarrow \pi^+ \mu^+ \mu^-) / \Gamma(K^+ \rightarrow \pi^+ e^+ e^-)$, which appears to be lower than the prediction obtained at leading order in the chiral expansion [4].

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Parity violating observables, such as the asymmetry in the polarization of the outgoing μ^+ and μ^- , are sensitive to short-distance dynamics. In the Standard Model (SM), the effect arises from the interference between the one-photon amplitude and one-loop Z -penguin and W -box Feynman diagrams [1,5,6]. It has been shown [1] that the muon polarizations can be predicted in terms of the well-known K_{l3} semileptonic decay form factors, and a parameter ξ that carries “clean” information (that means, relatively free from nonperturbative effects) on the quark masses and mixing angles. The explicit expression of ξ in terms of the quark mixing parameters has been calculated by Buchalla and Buras [6] up to next-to-leading order in QCD, where the dependence on the renormalization scale is shown to be significantly reduced.

The muon polarization asymmetry receives also potentially significant contributions from the interference of the one-photon amplitude with parity-violating Feynman diagrams in which the muon pair is produced by two-photon exchange. Though these contributions are difficult to evaluate—they arise from nonperturbative QCD—a detailed analysis performed in Ref. [7] seems to indicate that they are smaller than the short-distance contributions mentioned above. Here we will take this as an assumption, focusing our attention on the effects on the muon polarization arising from the short-distance part.

The theoretical analyses usually concentrate on the case of longitudinal muon polarizations. This is convenient for an obvious reason, which is the fact that the polarization direction is defined in each case by the muon momentum, and no external axes have to be introduced. However, the price one has to pay is that the so-defined longitudinal polarization asymmetry Δ_{long} is not a Lorentz-invariant magnitude, hence its value depends on the chosen reference frame. In the literature, it is usual to define Δ_{long} in the rest frame of the $\mu^+\mu^-$ pair, and to present the theoretical results in terms of the muon pair invariant mass, q^2 , and θ , the angle between the three-momenta of the kaon and the μ^- in this reference frame. Then, to compare the measurements in the LAB system with the theoretical predictions, it is necessary not only to measure the muon polarization, but also to reconstruct the full kinematics of each event in order to perform the corresponding boost to the $\mu^+\mu^-$ rest frame. In addition, cuts on the variable θ , which can improve the sensitivity to the short-distance parameter ξ mentioned above [7], do not translate, in general, into cuts on the pion and muon directions in the LAB. All these facts produce additional sources of uncertainties in the analysis. The aim of this work is to point out these difficulties, and propose the longitudinal polarization asymmetry defined in the LAB system, $\Delta_{long}^{(LAB)}$, as the best observable to be contrasted with experiment. We analyse here the kinematics for the process in the LAB frame, and calculate the expected sensitivity of $\Delta_{long}^{(LAB)}$ to the parameter ξ , for both stopped and in-flight decaying kaons. For a fixed energy of the K^+ , we show that the sensitivity of the observable can be improved by a convenient cut on the LAB muon energy. In addition, we perform a similar analysis for the decay $K_L \rightarrow \mu^+\mu^-$. The study of the muon polarization is also important in this case, since it can provide a new signal of CP violation [8]. From our analysis, it arises that the asymmetry $\Delta_{long}^{(LAB)}$ is partially diluted when the decaying kaons are in flight.

The paper is organized as follows: in Section II we study the sensitivity of $\Delta_{long}^{(LAB)}$ to the SM parameters for the process $K^+ \rightarrow \pi^+\mu^+\mu^-$, and calculate the dependence of the observable with the K^+ energy. Then, in Section III, we perform a similar analysis for the process $K_L \rightarrow \mu^+\mu^-$, in which the kinematics is simpler. In section IV we present our

conclusions. Details on the LAB frame kinematics and phase space integrations are given in the Appendix.

II. MUON POLARIZATION AND KINEMATICS FOR $K^+ \rightarrow \pi^+ \mu^+ \mu^-$

As stated, the decay rate for $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ is dominated by the one-photon exchange contribution, which is parity-conserving. The corresponding amplitude can be parametrized as

$$\mathcal{M}^{(PC)} = \frac{\alpha G_F \sin \theta_C}{\sqrt{2}} f(q^2) (p_K + p_\pi)^\mu \bar{u}(p_-, s_-) \gamma_\mu v(p_+, s_+), \quad (1)$$

where p_K , p_π and p_\pm are the four-momenta of the kaon, pion and μ^\pm respectively, and $q^2 = (p_+ + p_-)^2$ stands for the squared $\mu^+ \mu^-$ invariant mass. We consider the general case of polarized muons, being s_\pm the corresponding polarization vectors.

In the Standard Model, in addition to the dominant term (1), the decay amplitude contains a parity-violating piece. This can be written in general as

$$\mathcal{M}^{(PV)} = \frac{\alpha G_F \sin \theta_C}{\sqrt{2}} [B(p_K + p_\pi)^\mu + C(p_K - p_\pi)^\mu] \bar{u}(p_-, s_-) \gamma_\mu \gamma_5 v(p_+, s_+), \quad (2)$$

where the parameters B and C get contributions from both short- and long-distance physics. The short-distance contributions arise mainly from Z -penguin and W -box Feynman diagrams and carry clean information on the flavour structure of the SM. Hence, the experimental determination of B and C would be very interesting from the theoretical point of view, provided that the long-distance effects are under control. Since the total decay amplitude is dominated by the parity-conserving piece, to get this information one is lead to search for a parity-violating observable. The muon polarizations are immediate candidates in this sense.

It can be seen that, from the experimental point of view, the measurement of the μ^+ polarization is strongly favoured in comparison with that of the μ^- . The reason is that the μ^- give rise to the formation of muonic atoms when they are stopped in materials, and this makes it difficult to measure the polarization [9]. We will concentrate then in the polarization of the outgoing μ^+ , summing over the final μ^- states. In an arbitrary reference frame, the decay rate for polarized μ^+ is given by

$$\Gamma(s_+) = \frac{1}{2E_K} \int d\Phi \sum_{s_-} |\mathcal{M}(s_+)|^2, \quad (3)$$

where $d\Phi$ is the Lorentz-invariant differential phase space,

$$d\Phi = (2\pi)^4 \delta^{(4)}(p_K - p_\pi - p_+ - p_-) \prod_{a=\pi,+, -} \frac{d^3 p_a}{(2\pi)^3 2E_a}. \quad (4)$$

The polarization asymmetry of the outgoing μ^+ , in the direction given by s_+ , is defined now as

$$\Delta \equiv \frac{\Gamma(s_+) - \Gamma(-s_+)}{\Gamma(s_+) + \Gamma(-s_+)} . \quad (5)$$

As long as s_+ transforms as a four-vector, this quantity is clearly Lorentz-invariant. However, one has to take some care when referring to the *longitudinal* or *transverse* muon polarizations. A polarization vector that is parallel to the muon momentum in a particular frame, acquires in general a transverse component when one moves to a boosted system. Hence, in principle, the value of the longitudinal polarization asymmetry Δ_{long} obtained in an arbitrary reference frame may change, or even vanish, when the longitudinal polarization is measured in the LAB frame. The same can be applied to transverse polarizations, which also require the introduction of an additional reference plane (*e.g.* the plane of the decay, in the rest frame of the K^+).

It is usual (see for instance Refs. [1,6,7]) to define Δ_{long} in the rest frame of the $\mu^+\mu^-$ pair. As commented in the introduction, this represents a problem from the experimental point of view, since the theoretical prediction obtained for the asymmetry cannot be contrasted by measuring only the longitudinal muon polarizations in the LAB frame. One should instead fully reconstruct each observed event (not just look at the final μ^+), in order to boost all decay products to the particular frame proposed, and then perform the comparison with the theoretical value. Or, alternatively, one can boost the polarization vector s_+ , defined to be parallel to the μ^+ momentum in the rest frame of the $\mu^+\mu^-$ pair, to the LAB frame. Then, to compare with the theoretical prediction, one would have to measure the LAB μ^+ polarization along a different axis for each individual event, this axis being determined by the boost. Once again the analysis turns out to be quite involved.

Our proposal is simple: it just consists in considering an observable equivalent to (5), but defined directly in the LAB system. The advantage is that, once the process has been identified, the theoretical prediction can be contrasted just by analysing the final μ^+ polarization, without taking care about the energy and angular distribution of the remaining μ^- and π^+ . In particular, we can take s_+ to be longitudinal in the LAB system, and calculate the value for $\Delta_{long}^{(LAB)}$. As we show below, the result is in general different from that obtained in the $\mu^+\mu^-$ rest frame, and depends on the energy of the decaying K^+ .

The detailed calculation of the decay rate for polarized μ^+ in the LAB reference frame is presented in the Appendix. We end up with the following expression:

$$\Gamma(s_+) = \frac{(\alpha G_F \sin \theta_C)^2}{16\pi^2 E_K} \int_{E_{min}}^{E_{max}} dE_+ \int_{h(E_+)}^1 d(\cos \theta) |\vec{p}_+| [g_0(z) + (s_+ \cdot p_K) g_1(z)] , \quad (6)$$

where $z = (p_K \cdot p_+) = E_K E_+ - |\vec{p}_K| |\vec{p}_+| \cos \theta$, and the integration limits E_{min} , E_{max} and $h(E_+)$ are functions of the Lorentz factor γ characterising the boost from the K^+ rest frame to the LAB system ($\gamma = E_K/m_K$). The functions $g_{0,1}(z)$, given in the Appendix, carry the information on the form factors f , B and C introduced in Eqs. (1) and (2).

We concentrate on the longitudinal μ^+ polarization, $\Delta_{long}^{(LAB)}$, which can be trivially obtained from (5) and (6) by taking

$$s_+^\alpha = \frac{1}{m_\mu} \left(|\vec{p}_+| , \frac{E_+}{|\vec{p}_+|} \vec{p}_+ \right) , \quad (7)$$

with E_+ , \vec{p}_+ in the LAB frame. Now, in order to determine the sensitivity of this observable to the parameters of interest, we need some theoretical input for the form factors f , B and C . In the case of $f(q^2)$, which corresponds to the effective vertex $K\pi\gamma^*$, one can use the experimental information from the decay $K^+ \rightarrow \pi^+ e^+ e^-$. It is seen [10] that the absolute value of this form factor can be approximated by

$$|f(q^2)| = |f(0)| \left(1 + \lambda \frac{q^2}{m_\pi^2} \right), \quad (8)$$

with $|f(0)| = 0.294$ and $\lambda = 0.105$. On the other hand, from existing analyses within Chiral Perturbation Theory [4,7], one expects the imaginary part of $f(q^2)$ to be negligibly small compared with the real part.

In the case of the parity-violating amplitude, the situation is more complicated due to the interference between short- and long-distance contributions. As stated above, the long-distance effects arise from nonperturbative QCD and are very difficult to estimate [7]. To get definite numerical results, we will concentrate here only on the effect produced by the short-distance part (in fact, the estimates in Ref. [7] indicate that this should be the dominant one), in which the q^2 dependence of B and C can be obtained from semileptonic kaon decays. We have thus

$$B = f_+(q^2) \xi, \quad C = \frac{1}{2} f_-(q^2) \xi, \quad (9)$$

where $f_+(q^2)$ and $f_-(q^2)$ are the well-known form factors for K_{l3} decays. We will use here a standard parametrization [11], taking

$$f_\pm(q^2) = f_\pm(0) \left(1 + \lambda_\pm \frac{q^2}{m_\pi^2} \right), \quad (10)$$

with $f_+(0) = 0.99$, $\lambda_+ = 0.03$, $f_-(0) = -0.33$ and $\lambda_- = 0$. The novel information is contained in ξ , which can be calculated in the SM in terms of the quark masses and mixing angles. One has [1,7]

$$\xi \simeq -\tilde{\xi}_c + \left[\frac{V_{ts}^* V_{td}}{V_{us}^* V_{ud}} \right] \tilde{\xi}_t, \quad (11)$$

where V stands for the Cabibbo-Kobayashi-Maskawa matrix, and $\tilde{\xi}_c$ and $\tilde{\xi}_t$ arise from the contributions of Z -penguins and W -boxes. QCD corrections introduce some dependence on the renormalization scale, though this can be reduced with the inclusion of next-to-leading order contributions [6].

Here we just keep ξ as a parameter, and refer the reader to Ref. [6] for the detailed analysis of its explicit dependence on the quark masses and mixing angles. Since both B and C are linear in ξ , the muon longitudinal polarization asymmetry can be written as

$$\Delta_{long} = \pm \text{Re} \xi R, \quad (12)$$

where the \pm signs correspond to $|f(0)| = \pm f(0)$ respectively. To see the sensitivity of Δ_{long} to ξ , we concentrate on the value of the “kinematic” factor R , which can be computed numerically using the above inputs for the form factors.

We recall that Δ_{long} , and thus R , depend in general on the reference frame in which the polarization vectors are defined to be parallel to the μ^+ momenta. In the LAB system, R can be calculated by means of Eqs. (6) and (7) in terms of the energy of the decaying K^+ . The resulting curve is shown in Fig. 1. It can be seen that the asymmetry is maximized when the kaons are at rest, with $R \simeq -2.9$, while the effect turns out to be diluted for in-flight K^+ . For a dilation factor $\gamma \rightarrow \infty$ we end up with $R \simeq -1.6$.

In the case of high-energy kaons the sensitivity can be improved by performing a convenient cut in the μ^+ energy. We have analysed the situation for a dilation factor $\gamma = 12$, this means, a kaon energy of about 6 GeV. This is the energy of the K^+ beam in the experiment E865 at the BNL AGS, used to study K^+ decays involving three charged final particles, and suggested as one of the best candidates to perform the measurement of the μ^+ polarization in $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ [12]. The dependence of the asymmetry with the chosen range of the outgoing μ^+ energy for $\gamma = 12$ can be seen from Fig. 2, where we plot the differential rates

$$\frac{d\Gamma(s_+)}{dE_+} + \frac{d\Gamma(-s_+)}{dE_+}, \quad \frac{1}{\text{Re}\xi} \left(\frac{d\Gamma(s_+)}{dE_+} - \frac{d\Gamma(-s_+)}{dE_+} \right)$$

—the latter, up to a global sign— in terms of the μ^+ energy E_+ . As it is shown in the figure, there is a change of sign in the μ^+ polarization for $E_+ \sim 1$ GeV. This leads to a reduction in the value of $|R|$ when integrating over the whole range of μ^+ energies. By taking a lower cut at $E_+ = 1$ GeV, the asymmetry increases from $R \simeq -1.6$ to -2.1 , while the number of events gets reduced only by a factor 0.82.

Finally we notice that, by working in the $\mu^+ \mu^-$ rest frame, one obtains $|R| = 2.3$ [7]. Thus the best sensitivity for Δ_{long} , with no phase-space cuts, would be obtained from the decay of stopped kaons.

III. MUON POLARIZATION AND KINEMATICS FOR $K_L \rightarrow \mu^+ \mu^-$

The above discussion about the fermion polarizations and the dependence on the reference frame can be also applied to the decay $K_L \rightarrow \mu^+ \mu^-$. In this case, the longitudinal polarizations of the outgoing muons have also a considerable theoretical interest, since the measurement of nonzero polarizations would represent a new signal of CP violation [8]. It is clear that, being $K_L \rightarrow \mu^+ \mu^-$ a two-body decay, the kinematics is now much simpler than in the $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ case.

Using a similar notation as in the previous section, the decay amplitude for $K_L \rightarrow \mu^+ \mu^-$ can be written in terms of two parameters A and B ,

$$\mathcal{M} = \bar{u}(p_-, s_-) (i B + A \gamma_5) v(p_+, s_+). \quad (13)$$

To study this process, it is usual to work in the kaon rest frame, where the analysis is simpler. One can define the longitudinal μ^+ polarization asymmetry $\Delta_{long}^{(rest)}$ by using an expression similar to (5), and taking the polarization vectors \vec{s}_+ to be parallel to the μ^+ three-momenta in the kaon rest frame. As we have discussed below, $\Delta_{long}^{(rest)}$ will in general be different from $\Delta_{long}^{(LAB)}$, defined by taking \vec{s}_+ parallel to \vec{p}_+ in the LAB system, if the decaying K_L are in flight.

Let us analyse the dependence of $\Delta_{long}^{(LAB)}$ with the energy of the decaying K_L , $E_K = \gamma m_K$. As before, we sum over the final μ^- polarizations, obtaining

$$\sum_{s_-} |\mathcal{M}|^2 = m_K^2 (|A|^2 + \beta_0^2 |B|^2) + 4m_\mu \text{Im}(BA^*)(s_+ \cdot p_K), \quad (14)$$

where $\beta_0 = (1 - 4m_\mu^2/m_K^2)^{1/2}$, and the factor $\text{Im}(BA^*)$ carries the CP violation effects. For this process the integration over the phase space is straightforward, and we can work directly in the LAB system. The decay rate for polarized μ^+ is found to be

$$\Gamma(s_+) = \frac{m_K^2}{16\pi E_K |\vec{p}_K|} \int_{E_{min}}^{E_{max}} dE_+ \left[|A|^2 + \beta_0^2 |B|^2 + \frac{4m_\mu}{m_K^2} (s_+ \cdot p_K) \text{Im}(BA^*) \right], \quad (15)$$

where the limits of integration are

$$E_{min} = \frac{E_K - \beta_0 |\vec{p}_K|}{2}, \quad E_{max} = \frac{E_K + \beta_0 |\vec{p}_K|}{2}. \quad (16)$$

In the case of longitudinal polarization vectors, the scalar product in (15) is given by

$$(s_+ \cdot p_K) = \frac{1}{m_\mu |\vec{p}_+|} \left(\frac{E_+ m_K^2}{2} - E_K m_\mu^2 \right), \quad (17)$$

and we obtain for the total μ^+ polarization

$$\Delta_{long}^{(LAB)} = \frac{\text{Im}(BA^*)}{|A|^2 + \beta_0^2 |B|^2} \left[\frac{2\beta'}{\beta\beta_0} - \frac{(1 - \beta_0^2)}{\beta\beta_0} \log \left(\frac{1 + \beta'}{1 - \beta'} \right) \right], \quad (18)$$

where

$$\beta = \sqrt{1 - \gamma^{-2}}, \quad \beta' = \begin{cases} \beta & , \quad \gamma < \frac{m_K}{2m_\mu} \\ \beta_0 & , \quad \gamma \geq \frac{m_K}{2m_\mu} \end{cases}. \quad (19)$$

Notice that the dependence of the observable in Eq. (18) with the K_L energy is contained into the factor in square brackets, hence it does not depend on the dynamics. As in the case of $K^+ \rightarrow \pi^+ \mu^+ \mu^-$, it is seen that the asymmetry is reduced when the kaons are more energetic, though the effect is rather small. In the limit $\gamma \rightarrow \infty$, the longitudinal polarization is reduced by a factor

$$r \equiv \frac{\Delta_{long}^{(\gamma \rightarrow \infty)}}{\Delta_{long}^{(rest)}} = \frac{1}{\beta_0} - \frac{(1 - \beta_0^2)}{2\beta_0^2} \log \left(\frac{1 + \beta_0}{1 - \beta_0} \right) \simeq 0.77. \quad (20)$$

Still this ratio can be slightly increased by taking into account the energy distribution of the muons in the LAB system. Since the CP-conserving terms in Eq. (14) are independent of the kinematic variables, the dependence of the μ^+ polarization with E_+ is basically given by the scalar product (17). For $E_K > m_K^2/(2m_\mu)$, it is seen that the polarization changes sign at $E_+ = E_0 \equiv 2E_K m_\mu^2/m_K^2$, thus the sensitivity can be improved by making a lower cut on E_+ . Taking *e.g.* $E_+ \geq 2E_0$, one gets $r = 0.89$, while the number of events is reduced by about 15%.

IV. CONCLUSIONS

We have analysed the decays $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ and $K_L \rightarrow \mu^+ \mu^-$, focusing our attention on the longitudinal polarization of the outgoing μ^+ . For these processes, the asymmetry in the production of muons with opposite helicities has a significant theoretical interest in connection with the flavour mixing and the structure of the Standard Model.

The longitudinal polarization asymmetry Δ_{long} depends in general on the chosen reference frame, since the helicity of a massive particle can change after a Lorentz transformation. Here we have considered $\Delta_{long}^{(LAB)}$, that means, the longitudinal polarization asymmetry defined in the laboratory system. For the decay $K^+ \rightarrow \pi^+ \mu^+ \mu^-$, the advantage of choosing this frame is that the theoretical predictions can be contrasted with experiment just by measuring the polarization of the outgoing μ^+ , summing over all energies and angular distributions of the remaining μ^- and π^+ .

For both processes, we analyse the dependence of $\Delta_{long}^{(LAB)}$ with the energy of the decaying kaons, showing that the asymmetry is partially diluted when the kaons are in flight. In the case of the decay $K^+ \rightarrow \pi^+ \mu^+ \mu^-$, this is illustrated by the curve in Fig. 1 (we have neglected here long-distance contributions arising from two-photon exchange). We have considered in particular the case of in-flight kaons with energy of 6 GeV. For this energy, it is shown that there is a change of sign in the μ^+ polarization for μ^+ energies of about 1 GeV, thus a lower energy cut at this point allows to improve the asymmetry. On the other hand, in the case of the decay $K_L \rightarrow \mu^+ \mu^-$ it is shown that the dilution is purely kinematic, *i.e.* it does not depend at all on the dynamics of the process. In the limit of large K_L energies, the asymmetry $\Delta_{long}^{(LAB)}$ is found to be reduced by about 23% with respect to the value obtained when the decaying kaons are at rest.

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APPENDIX:

We calculate here the decay rate $\Gamma(s_+)$ for the process $K^+ \rightarrow \pi^+ \mu^+ \mu^-$ in the LAB reference frame. From Eqs. (1), (2) and (3), we have

$$\Gamma(s_+) = \frac{(\alpha G_F \sin \theta_C)^2}{2E_K} \int d\Phi [F_0 + (s_+ \cdot T)] , \quad (A1)$$

where

$$\begin{aligned}
F_0 &= |f(q^2)|^2 [2(2z - q^2)(m_K^2 - 2z) - 4zm_\pi^2], \\
T^\mu &= \text{Re}(f(q^2)B^*) \left[(2m_K^2 - 2m_\pi^2 + q^2 - 4z)p_K^\mu + 2(z - m_K^2)p_-^\mu \right] \\
&\quad + \text{Re}(f(q^2)C^*)(q^2 p_K^\mu - 2zp_-^\mu),
\end{aligned} \tag{A2}$$

with $z \equiv (p_K \cdot p_+)$.

Let us first perform the integration over the μ^- and π^+ phase space variables. To do this, we write the differential phase space as

$$d\Phi = \frac{d^3 p_+}{(2\pi)^3 2E_+} d\Phi',$$

with

$$d\Phi' = (2\pi)^4 \delta^{(4)}(p_K - p_\pi - p_+ - p_-) \frac{d^3 p_-}{(2\pi)^3 2E_-} \frac{d^3 p_\pi}{(2\pi)^3 2E_\pi}.$$

Notice that F_0 is a function of the invariants q^2 and z . Then the integral over the μ^- and π^+ momenta must be a function of z only,

$$g_0(z) = \int F_0(z, q^2) d\Phi'. \tag{A3}$$

In the same way, for the second term in the integrand of (A1) we can write

$$\int T^\mu d\Phi' = g_1(z) p_K^\mu + g_2(z) p_+^\mu. \tag{A4}$$

Since s_+ is by definition orthogonal to p_+ , the function $g_2(z)$ does not contribute to the decay rate (A1) and we only need to compute $g_1(z)$. The latter can be written as

$$g_1(z) = \int F_1(z, q^2) d\Phi', \tag{A5}$$

where the integrand is given by

$$F_1(z, q^2) = \frac{[z(p_+ \cdot T) - m_\mu^2(p_K \cdot T)]}{z^2 - m_\mu^2 m_K^2}. \tag{A6}$$

To perform the integrals in (A3) and (A5) explicitly, one can choose a convenient reference frame. Let us consider the system in which the kaon and the μ^+ three-momenta have equal magnitude and direction:

$$\vec{p}_K - \vec{p}_+ = \vec{p}_- + \vec{p}_\pi = 0. \tag{A7}$$

Denoting by y the cosine of the angle between the K^+ and π^+ directions in this frame, the functions $g_{0,1}(z)$ can be obtained from

$$g_i(z) = \int F_i(z, q^2) d\Phi' = \frac{1}{16\pi} \frac{[(m_K^2 - m_\pi^2 - 2z)^2 - 4m_\mu^2 m_\pi^2]^{1/2}}{m_K^2 + m_\mu^2 - 2z} \int_{-1}^1 F_i(z, q^2) dy, \tag{A8}$$

where the $\mu^+\mu^-$ invariant mass q^2 is given in terms of z and y by

$$q^2 = m_K^2 + m_\pi^2 - \frac{(m_K^2 - z)(m_K^2 + m_\pi^2 - 2z)}{m_K^2 + m_\mu^2 - 2z} + \frac{(z^2 - m_K^2 m_\mu^2)^{1/2} [(m_K^2 - m_\pi^2 - 2z)^2 - 4m_\pi^2 m_\mu^2]^{1/2}}{m_K^2 + m_\mu^2 - 2z} y. \quad (\text{A9})$$

The advantage of choosing this particular reference frame is that the integration limits for y do not depend on the K^+ or μ^+ momenta. These only enter through the Lorentz invariant product $z = (p_K \cdot p_+)$, which does not depend on y .

Being $g_0(z)$ and $g_1(z)$ Lorentz-invariant functions, we can move now easily to the LAB reference frame. The total $K^+ \rightarrow \pi^+\mu^+\mu^-$ decay rate for polarized μ^+ will be given by

$$\begin{aligned} \Gamma(s_+) &= \frac{1}{2E_K} (\alpha G_F \sin \theta_C)^2 \int \frac{d^3 p_+}{(2\pi)^3 2E_+} [g_0(z) + (s_+ \cdot p_K) g_1(z)] \\ &= \frac{(\alpha G_F \sin \theta_C)^2}{16\pi^2 E_K} \int_{E_{min}}^{E_{max}} dE_+ \int_{h(E_+)}^1 d(\cos \theta) |\vec{p}_+| [g_0(z) + (s_+ \cdot p_K) g_1(z)] \end{aligned} \quad (\text{A10})$$

where θ stands for the angle between the μ^+ and the kaon in the LAB system, and z is given by

$$z = E_K E_+ - |\vec{p}_K| |\vec{p}_+| \cos \theta. \quad (\text{A11})$$

The limits of integration in (A10) are found to be

$$\begin{aligned} E_{min} &= \begin{cases} m_\mu & , \quad \gamma \leq \frac{E_0}{m_\mu} \\ \gamma E_0 - \gamma \beta |\vec{p}_0| & , \quad \gamma > \frac{E_0}{m_\mu} \end{cases} \\ E_{max} &= \gamma E_0 + \gamma \beta |\vec{p}_0| \\ h(E_+) &= \begin{cases} -1 & , \quad m_\mu \leq E_+ \leq \gamma E_0 - \gamma \beta |\vec{p}_0| \\ \frac{\gamma E_+ - E_0}{\gamma \beta \sqrt{E_+^2 - m_\mu^2}} & , \quad \gamma E_0 - \gamma \beta |\vec{p}_0| < E_+ \leq \gamma E_0 + \gamma \beta |\vec{p}_0| \end{cases} \end{aligned} \quad (\text{A12})$$

where γ and β are the Lorentz dilation factor and the velocity of the decaying kaon respectively,

$$\gamma = \frac{E_K}{m_K}, \quad \beta = \sqrt{1 - \gamma^{-2}} = \frac{|\vec{p}_K|}{E_K}, \quad (\text{A13})$$

and $E_0, |\vec{p}_0|$ are defined as

$$E_0 = \frac{m_K}{2} \left(1 - \frac{m_K^2}{m_\pi^2} \right) - \frac{m_\mu m_K}{m_\pi}, \quad |\vec{p}_0| = \sqrt{E_0^2 - m_\mu^2}. \quad (\text{A14})$$

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FIGURES

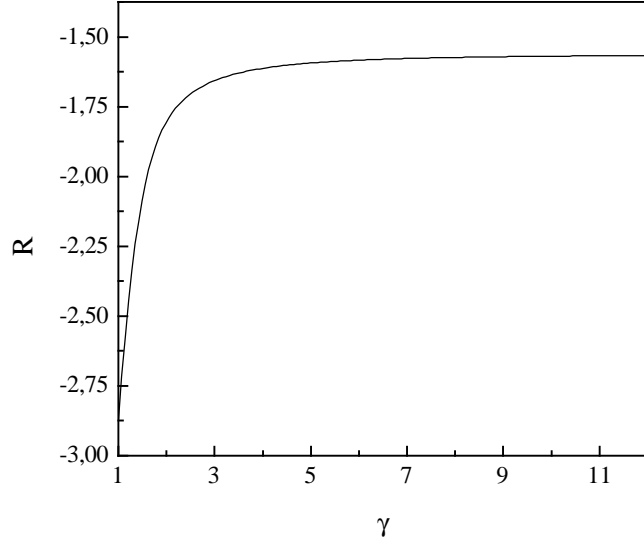


FIG. 1. Asymmetry parameter R , as function of the Lorentz factor γ .

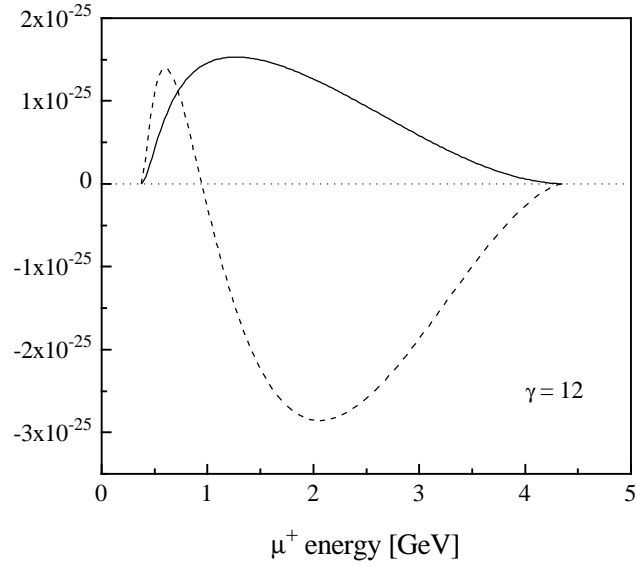


FIG. 2. Differential decay rates for the process $K^+ \rightarrow \pi^+ \mu^+ \mu^-$, as functions of the energy of the final μ^+ , for a Lorentz factor $\gamma = 12$. The solid line stands for the total width, while the dashed one corresponds —up to a global sign— to the difference between the rates for opposite longitudinal μ^+ polarizations, scaled by a factor $(\text{Re}\xi)^{-1}$.